UNIVERSAL PI-ALGEBRAS AND ALGEBRAS OF GENERIC MATRICES †

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ABSTRACT

Let $\Omega[\xi]$ denote the polynomial algebra (with 1) in commutative indeterminates $\xi_{ij}^{(k)}$, $1 \leqq i, j \leqq n$, $1 \leqq k < \infty$, over a commutative ring Ω. The *algebra of generic matrices* $\Omega[Y]$ is defined to be the Ω-subalgebra of $M_n(\Omega[\xi])$ generated by the matrices $Y_k = (\xi_{ij}^{(k)})$, $1 \leqq i,j \leqq n$, $1 \leqq k < \infty$. This algebra has been studied extensively by Amitsur and by Procesi [8]; in particular Amitsur [1] has used it to construct a finite dimensional, central division algebra $\Omega(Y)$ which is not a crossed product. In this paper we shall prove, for Ω a domain, that $\Omega(Y)$ has exponent n in the Brauer group (Amitsur may already know this fact); consequently, for Ω an infinite field and n a multiple of 4, if $f(X_1, \cdots, X_m)$ is a polynomial linear in all the X_i but one (similar to Formanek's central polynomials for matrix rings) and f^2 is central for $M_n(\Omega)$, then f is central for $M_n(\Omega)$. (The existence of a polynomial not central for $M_n(\Omega)$, but whose square is central for $M_n\Omega$) is equivalent to every central division algebra of degree n containing a quadratic extension of its center; well-known theory immediately shows this is the case of $4 \mid n$ and $8 \nmid n$.) Also, information is obtained about $\Omega(Y)$ for arbitary Ω, most notably that the Jacobson radical is the set of nilpotent elements.

1. Preliminaries

All rings and algebras are associative with 1. Let Ω be a commutative ring. We shall deal with the category of Ω-algebras, hereafter called *algebras*. Consider $\Omega\{X\} \equiv \Omega\{X_1, X_2, \cdots\}$, the free algebra generated by a countable set of non-commuting indeterminates over Ω. The elements of $\Omega\{X\}$ are called polynomials. An element f of $\Omega\{X\}$ contained in the subalgebra generated by $X_1, \cdots, X_m$ is written $f(X_1, \cdots, X_m)$. Note that $\Omega\{X\}$ is free as a Ω-module, with countable base

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consisting of 1 and the distinct monomials $X_{i_1} \cdots X_{i_t}$, $i_j = 1, 2, \cdots$. Writing a polynomial f as a unique linear combination of elements of this base, we call the monomials in this expression the *monomials of* f; f is *linear in* X_i if X_i has degree 1 in each monomial of f.

The *standard polynomial* $S_{2n}(X_1, \cdots, X_{2n}) \equiv \Sigma_\pi (\operatorname{sg} \pi) X_{\pi 1} \cdots X_{\pi(2n)}$, π a permutation of $(1, \cdots, 2n)$. A polynomial f is an *identity of* R if f is in the kernel of every homomorphism from $\Omega\{X\}$ to R; R is a PI-*algebra* if $(S_{2n})^m$ is an identity of R for suitable n, m (the smallest such n is called the *degree of* R). A consequence of the existence of central polynomials for matrix algebras over commutative rings is the following theorem.

THEOREM A. ([9].) *Any nonzero ideal of a semiprime* PI-*algebra intersects the center nontrivially.*

Let R be an algebra with center C. A *submonoid of* C is a multiplicative set with 1. Given a submonoid S of C, let R_S be the (commutative) localization of R (as C-module) with respect to S, given the natural algebra structure by defining $(r_1 s_1^{-1})(r_2 s_2^{-1}) = (r_1 r_2)(s_1 s_2)^{-1}$. It is easy to see (refer to [6, Lem. 4.2, p. 26]) that for any subring H of C containing S, the algebra homomorphism

$$R_S \to R \underset{H}{\otimes} H_S,$$

given by $rs^{-1} \mapsto r \otimes s^{-1}$, is an isomorphism. An extensive treatment of this procedure, known as *central localization*, is given for PI-algebras in [10], and one application is a fast proof of the following fact, due to many people.

THEOREM B. ([9].) *Let R be a prime* PI-*algebra of degree n and let $S = C - \{0\}$. Then R_S is a simple artinian* PI-*algebra of degree n,* cent $R_S = C_S$, *the quotient field of C, and R_S is the classical left and right quotient algebra of R.*

Now for any s in S, let $\langle s \rangle$ denote the submonoid $\{s^j \mid j = 0, 1, 2, \cdots\}$ and let $\nu_s \colon R \to R_{\langle s \rangle}$ be the canonical homomorphism $r \mapsto r1^{-1}$. $\mathcal{S} = \{R_{\langle s \rangle} \mid s \in S\}$ has a partial ordering given by $R_{\langle s \rangle} \leqq R_{\langle s' \rangle}$ if $\nu_{s'}(s)$ is invertible in $R_{\langle s' \rangle}$; in this case there is a unique homomorphism $\phi_{s,s'} \colon R_{\langle s \rangle} \to R_{\langle s' \rangle}$ such that $\phi_{s,s'}\nu_s = \nu_{s'}$. $(\mathcal{S}, \{\phi_{s,s'}\})$ clearly has the direct limit R_S (refer to [4, pp. 87–88]). This fact will be used in Section 3.

2. Basic properties of algebras of generic matrices

An ideal W of an algebra R is a *T-ideal* of R if W is invariant under all endomorphisms of R. $\Omega\{X\}/W$ is a *universal* PI-*algebra* if W is a T-ideal of $\Omega\{X\}$

containing $(S_{2n}(X_1, \cdots, X_{2n}))^m$ for some n, m. Note that for any PI-algebra R, the set W of identities of R is a T-ideal of $\Omega\{X\}$ containing some $(S_{2n}(X_1, \cdots, X_{2n}))^m$, so $\Omega\{X\}/W$ is a universal PI-algebra.

Now let $\Omega[\xi] \equiv \Omega[\xi_{ij}^{(k)}]$ be the polynomial algebra in commutative indeterminates $\xi_{ij}^{(k)}$ over Ω, $1 \leqq i, j \leqq n$, $1 \leqq k < \infty$. Fixing n, we define the *algebra of generic* $n \times n$ *matrices* $\Omega\{Y\}$ as the subalgebra of $M_n(\Omega[\xi])$ generated by the generic matrices $Y_k = (\xi_{ij}^{(k)})$, $1 \leqq i, j \leqq n$, $1 \leqq k < \infty$. Note, for any commutative algebra K and elements $r_1, r_2, \cdots$ of $M_n(K)$, there is a unique Ω-algebra homomorphism of $\Omega\{Y\}$ to $M_n(K)$ sending $Y_1 \mapsto r_1$, $Y_2 \mapsto r_2, \cdots$. For, if $r_k = (r_{ij}^{(k)}), r_{ij}{}^{(k)}$ in K, then the Ω-algebra homomorphism $\Omega[\xi] \to K$ sending $\xi_{ij}^{(k)} \to r_{ij}^{(k)}$, $1 \leqq i, j \leqq n$, $1 \leqq k < \infty$, extends uniquely to an Ω-algebra homomorphism of $M_n(\Omega[\xi])$ to $M_n(K)$; the restriction to $\Omega\{Y\}$ is the required homomorphism.

Let Ω' be the classical ring of quotients of $\Omega[\xi]$; if Ω is a domain then Ω' is an infinite field. The following theorem is an extension of a well-known result (when Ω is a field).

THEOREM 1. (i). *$\Omega\{Y\} \approx \Omega\{X\}/W$ where W is the set of identities of $M_n(\Omega')$ (as Ω-algebra). Hence, $\Omega\{Y\}$ is a universal* PI-*algebra.*

(ii). $\Omega\{Y\}\Omega' = M_n(\Omega')$.

PROOF. (i). Let $\phi: \Omega\{X\} \to \Omega\{Y\}$ be the epimorphism sending the indeterminate X_k into the generic matrix Y_k. Since $\Omega\{Y\} \subseteq M_n(\Omega')$, we have $W \subseteq \ker \phi$. On the other hand, suppose $f(X_1, \cdots, X_m) \in \ker \phi$. Then $f(Y_1, \cdots, Y_m) = 0$. Given $r_1, \cdots, r_m$ in $M_n(\Omega')$ one can find a homomorphism of $\Omega\{Y\}$ to $M_n(\Omega')$ sending $Y_1 \to r_1, \cdots, Y_m \to r_m$; hence $f(r_1, \cdots, r)_m = 0$ for all $r_1, \cdots, r_m$ in $M_n(\Omega')$, implying $f \in W$. Therefore $\ker \phi = W$.

(ii). For $1 \leqq i < n$, let $u = (i-1)n + j$ and write e_u for e_{ij}, ξ_{uk} for $\xi_{ij}^{(k)}$. Then $Y_k = \Sigma \xi_{uk} e_u$, $1 \leqq u, k \leqq n^2$. Let Z be the $n^2 \times n^2$ matrix whose (u, k) entry is $\xi_{u,k}$, $1 \leqq u$, $k \leqq n^2$. Note that each of the n^4 entries in Z is a distinct indeterminate. The theorem is proved if Z is invertible in $M_{n_2}(\Omega')$, for in that case we can express the elementary matrices e_u as linear combinations of $Y_1, \cdots, Y_k$. In view of the formula $\det Z = Z \operatorname{adj} Z$ (where *adj* means *adjoint matrix*), it suffices to show $\det Z$ is invertible in Ω'. Note this is trivial if Ω' is a field, because $\det Z \neq 0$.

In general, it suffices to prove $\det Z$ is not a zero-divisor in $\Omega[\xi]$. Le $\mathscr{S} = \{\xi_{ij}^{(k)} \mid 1 \leqq i, j \leqq n, 1 \leqq k < \infty\}$. We claim in fact for any $m > 0$ and any

$m \times m$ matrix A with distinct entries in $\mathscr{S}$, that $\det A$ is regular in $\Omega[\xi]$. Indeed, the assertion is immediate for $m = 1$, so we proceed by induction on m. Let $\mathscr{S}' = \{\text{entries of } A\}$; by assumption $\mathscr{S}'$ has cardinality m^2. Write an element ω of $\Omega[\xi]$ as $\omega(\xi_1, \cdots, \xi_t)$ if $\xi_1, \cdots, \xi_t$ are the only indeterminates (in $\mathscr{S}$) occurring in ω. If $\det A$ is a zero-divisor in $\Omega[\xi]$, choose $\omega(\xi_1, \cdots, \xi_t)$ in $\Omega[\xi]$ with t minimal such that $\omega \det A = 0$. Obviously the coefficient of ξ_t^j in $\omega \det A$ is 0, each j. Write $\omega = \sum_{i=u}^{v} \omega_i(\xi_1, \cdots, \xi_{t-1})\xi_t^i$, suitable u, v, with $\omega_v \neq 0$. By assumption on t, $\omega_v \det A \neq 0$.

If $\xi_t \notin \mathscr{S}'$ then, by looking at the coefficient of ξ_t^v, we obtain $\omega_v \det A = 0$, contrary to the assumption. So $\xi_t \in \mathscr{S}'$. But, looking at the coefficient of ξ_t^{v+1} we obtain $\omega_v \det A' = 0$, where A' is the minor of A taken at the position of ξ_t. This contradicts the induction hypothesis because A' is an $(m-1) \times (m-1)$ matrix with distinct entries in $\mathscr{S}$. Hence $\det A$ is not a zero-divisor in $\Omega[\xi]$, yielding the claim and proving the theorem. Q.E.D.

An immediate consequence of Theorem 1 is that $\Omega\{Y\}$ is prime if and only if Ω is a domain.

3. Results when Ω is a domain

Assume throughout this section that Ω is a domain. As we have already noted, Ω' is then an infinite field and $\Omega\{Y\}$ is prime. Let $C' = \operatorname{cent}\Omega\{Y\}$. By Theorem B, we can form $\Omega(Y) = \Omega\{Y\}_{C'-[0]}$, a simple artinian algebra of degree n, whose center $\hat{C}$ is the quotient field of C'. Moreover, since $C' \subseteq \Omega[\xi]$, clearly we can view $\Omega(Y) \subseteq M_n(\Omega')$, whereby Theorem 1 implies $\Omega(Y)\Omega' = M_n(\Omega')$.

We shall often rely on the existence (demonstrated in characteristic 0 by Brauer [5] and in general by Amistur [2]) of a division Ω'-algebra D of degree n and exponent n in the Brauer group.

Amistur has proved, using this fact, that $\Omega(Y)$ is a division algebra. A similar proof using central localization, is as follows. Choose any nonzero $f(Y_1, \cdots, Y_m)$ in $\Omega\{Y\}$. Then $f(X_1, \cdots, X_m)$ is not an identity of D, so $f(d_1, \cdots, d_m) \neq 0$ for suitable $d_1, \cdots, d_m$ in D. But D has no nonzero nilpotent elements; hence, specializing $Y_1 \mapsto d_1, \cdots, Y_m \mapsto d_m$, we see $f(Y_1, \cdots, Y_m)$ is not nilpotent in $\Omega\{Y\}$. Hence, no nonzero element of $\Omega\{Y\}$ is nilpotent, and it follows that $\Omega(Y)$ has no nonzero nilpotent elements. Therefore $\Omega(Y)$ is a division algebra by the Wedderburn structure theorem. Incidentally, it clearly follows for an infinite domain Ω that if f^t is an identity of $M_n(\Omega)$ then f is also an identity.

Amitsur [1] has shown in characteristic 0 that $\mathbb{Q}(Y)$ is not a crossed product if n is divisible by 8 or by the square of an odd prime; Schacher and Small [11] have used Amitsur's methods to obtain similar results in characteristic $\neq 0$.

THEOREM 2. *$\Omega(Y)$ has exponent n in the Brauer group.*

PROOF. For any algebra R with arbitrary center C, define inductively $R^1 = R$ and $R^t = R^{t-1} \otimes_C R$. For example, $(\Omega\{X\})^t = \Omega\{X\} \otimes_\Omega \cdots \otimes_\Omega \Omega\{X\}$. The crux of the proof lies in the following lemma.

LEMMA. *If $\Omega(Y)^t$ has a set of u^2 matric units then, for any simple Ω'-algebra A of degree n, A^t also has a set of u^2 matric units.*

Given the lemma, the theorem is immediate. Indeed, let us assume $\Omega(Y)$ has exponent $t < n$ in the Brauer group, and let $\hat{C} = \operatorname{cent} \Omega(Y)$. Then $(\Omega(Y))^t \approx M_{n^t}(\hat{C})$ so $(\Omega(Y))^t$ has a set of $(n^t)^2$ matric units. Hence, by the lemma, A^t also has a set of $(n^t)^2$ matric units, A any simple Ω'-algebra. It follows A^t is a matrix algebra, so each A has exponent less then or equal to t in the Brauer group. This contradicts the existence of a division Ω'-algebra of degree n, having exponent Ω in the Brauer group; hence $\Omega(Y)$ must have degree n.

So it suffices to prove the lemma. Since $\hat{C}$ is the quotient field of C, we have $\Omega(Y) \approx \Omega\{Y\} \otimes_{C'} \hat{C}$, so $\Omega(Y)^t \approx \Omega\{Y\}^t \otimes_{C'} \hat{C}$. Viewing $C' \subseteq \operatorname{cent} \Omega\{Y\}^t$ via $c \mapsto 1 \otimes \cdots \otimes 1c$, c in C', let $R = \Omega\{Y\}^t$ and $S = C' - \{0\}$; then $R_S \approx \Omega\{Y\}^t \otimes_{C'} \hat{C}$ $\approx \Omega(Y)^t$.

Suppose $\{e_{ij} \mid 1 \leqq i,j \leqq u\}$ is a set of matric units for $\Omega(Y)^t$; that is, $e_{ij}e_{vw} = 0$ for $j \neq v$, $e_{ij}e_{jw} = e_{iw}$, and $\sum_{i=1}^u e_{ii} = 1$. Viewed in R_S, $e_{ij} = f_{ij}s^{-1}$, $1 \leqq i,j \leqq u$ f_{ij} in R, s in S. Let $v_S \colon R \to R_S$ be the canonical homomorphism $r \mapsto r1^{-1}$, and let $B = \ker v_S$. Then $f_{ij}f_{vw} \in B$ for $j \neq v$, $f_{ij}f_{jw} - f_{iw}s \in B$, and $\sum_{i=1} f_{ii} - s \in B$. Since these are a finite number of conditions and since R_S is the direct limit of the $R_{\langle c \rangle}$, c in S, there exists c in S such that, for $B' = \ker v_c \colon R \to R_{\langle c \rangle}$, $f_{ij}f_{vw} \in B'$ (for $j \neq v$), $f_{ij}f_{jw} - f_{iw}s \in B'$, and $\sum_{i=1}^u f_{ii} - s \in B'$. But $cs \neq 0$ in C', so for any simple algebra A of degree n, there exists a specialization $\psi \colon \Omega\{Y\} \to A$ such that $\psi(cs) \neq 0$; ψ induces a homomorphism

$$\hat{\psi} \colon R = \Omega\{Y\} \underset{C'}{\otimes}, \cdots, \underset{C'}{\otimes} \Omega\{Y\} \xrightarrow{\psi \otimes \cdots \otimes \psi} A \underset{C'}{\otimes} \cdots \underset{C'}{\otimes} A \to A \underset{\text{cent } A}{\otimes} \cdots \underset{\text{cent } A}{\otimes} A = A^t.$$

Clearly $\hat{\psi}(cs) \neq 0$, so $\hat{\psi}(cs)^{-1} \in \operatorname{cent} A$, implying $\hat{\psi}(c)^{-1} \in \operatorname{cent} A$; hence $\hat{\psi}$ induces a homomorphism $\hat{\psi} \colon R_{\langle c \rangle} \to A^t$ (given by $\hat{\psi}(rc^{-k}) = \hat{\psi}(r)\hat{\psi}(c)^{-k}$.)

For any r in R, let $\bar{r}$ denote $(\hat{\psi} \circ v_c)(r)$. Then $\bar{f}_{ij}\bar{f}_{vw} = 0$ for $j \neq v$, $\bar{f}_{ij}\bar{f}_{jw}\bar{s} - \bar{f}_{iw} = 0$

$\Sigma_{i=1}^{u} \bar{f}_{ii} = \bar{s}$, and $\bar{s}^{-1}$ exists. Hence $\{f_{ij}s^{-1} \mid 1 \leqq i,j \leqq u\}$ is a set of matric units for $\Omega(Y)^t$, proving the lemma (and thus the theorem). Q.E.D.

EXAMPLE 1. Let $n = 2, \Omega = \mathbb{Z}$, and $t = 2$. Then $\mathbb{Z}\{Y\}^2$ is not torsion-free as a module over its center, and is not prime. Indeed, the canonical injection $\phi: \mathbb{Z}\{Y\} \to \mathbb{Z}(Y)$ induces a homomorphism

$$\hat{\phi}: \mathbb{Z}\{Y\}^2 = \mathbb{Z}\{Y\} \otimes_{C'} \mathbb{Z}\{Y\} \xrightarrow{\phi \otimes \phi} \mathbb{Z}(Y) \otimes_{C'} \mathbb{Z}(Y) \to \mathbb{Z}(Y)^2,$$

and both assertions will follow from Proposition 1.

PROPOSITION 1. $\ker \hat{\phi} \neq 0$.

PROOF. We shall show in fact that

$$r = [Y_1^2, Y_2] \otimes_{C'} [Y_1, Y_3] - [Y_1, Y_2] \otimes_{C'} [Y_1^2, Y_3]$$

s a nonzero element of $\ker \hat{\phi}$, where $[x, y] \equiv xy - yx$.

First observe in the simple algebra $\mathbb{Z}(Y)$ that $Y_1^2 - Y_1 \operatorname{tr} Y_1 + \det Y_1 = 0$. Hence, $[Y_1^2, Y_2] - [Y_1, Y_2] \operatorname{tr} Y_1 + 0 = 0$, so $[Y_1^2, Y_2] = [Y_1, Y_2] \operatorname{tr} Y_1$. Therefore,

$$[Y_1^2, Y_2] \otimes_{\hat{C}} [Y_1, Y_3] = [Y_1, Y_2] \operatorname{tr} Y_1 \otimes_{\hat{C}} [Y_1, Y_3] = [Y_1, Y_2] \otimes_{\hat{C}} [Y_1, Y_3] \operatorname{tr} Y_1$$
$$= [Y_1, Y_2] \otimes_{\hat{C}} [Y_1, Y_3]$$

(where $\hat{C} = \operatorname{cent} \mathbb{Z}(Y)$), implying $r \in \ker \hat{\phi}$.

The proof $r \neq 0$ is a matter of comparing degrees. Suppose $r = 0$. Then the formal construction of tensor product shows that in $\mathbb{Z}\{Y\} \times \mathbb{Z}\{Y\}$, the algebra formally generated by ordered pairs (f, g), f, g in $\mathbb{Z}\{Y\}$, the element

$$([Y_1^2, Y_2], [Y_1, Y_3]) - ([Y_1, Y_2], [Y_1^2 Y_3])$$

is of the form

$$\sum_i ((f_{1i} + f_{2i}, g_i) - (f_{1i}, g_i) - (f_{2i}, g_i)) + \sum_j ((f_j, g_{1j} + g_{2j}) - (f_j, g_{1j}) - (f_j, g_{2j}))$$
$$+ \sum_k ((f_k, c_k g_k) - (c_k f_k, g_k))$$

where $c_k \in C' = \operatorname{cent} \mathbb{Z}\{Y\}$ and all the f and g are in $\mathbb{Z}\{Y\}$.

For any f in $\mathbb{Z}\{Y\}$, let $f^{(t)}$ denote the sum of those monomials of f of total degree t and let $(\Sigma(f_i, g_i))^{(s,t)}$ be defined as $\Sigma(f_i^{(s)}, g_i^{(t)})$. We claim $(\Sigma_k((f_k, c_k g_k) - (c_k f_k, g_k)))^{(2,3)} = 0$. Indeed, consider the polynomial $c_k(Y_1, \cdots, Y_m)$ in C'. Clearly $c_k(X_1, \cdots, X_m)$ is a central polynomial of $M_2(\mathbb{Z})$, so $[X_{m+1}, c_k(X_1, \cdots X_m)]$ is an identity of $M_2(\mathbb{Z})$. Since $\mathbb{Z}$ has characteristic 0, the standard Vandermonde argument shows that $[X_{m+1}, c_k(X_1, \cdots, X_m)]$ is the sum of identities of the form $[X_{m+1}, c_{ku}(X_1, \cdots, X_m)]$, homogeneous in each indeterminate. Hence each c_{ku} is an identity or a central polynomial of $M_2(\mathbb{Z})$, homogeneous in each indeterminate; clearly $c_{ku}(Y_1, \cdots, Y_m) \in C'$, and $c_k(Y_1, \cdots, Y_m) = \Sigma c_{ku}(Y_1, \cdots, Y_m)$. But any nonzero c_{ku} has total degree greater than or equal to 4 because $[X_{m+1}, c_{ku}(X, \cdots, X_m)]$ is an identity of $M_2(\mathbb{Z})$, not a multiple of the standard identity S_4 (refer to Amitsur-Levitzki [3]). Thus $(f_k, c_k g_k)^{(2,3)} = 0$ and $(c_k f_k, g_k)^{(2,3)} = 0$ for all k, as claimed.

Thus

$$
\begin{aligned}
([Y_1, Y_2], [Y_1^2, Y_3]) &= (([Y_1^2, Y_2], [Y_1, Y_3]) - ([Y_1, Y_2], [Y_1^2, Y_3]))^{(2,3)} \\
&= \Big(\sum_i ((f_{1i}+f_{2i}, g_i) - (f_{1i}, g_i) - (f_{2i}, g_i)) + \sum_j ((f_j, g_{1j} + g_{2j}) - (f_j, g_{1j}) - (f_j, g_{2j})) \\
&\quad + \sum_k ((f_k, c_k g_k) - (c_k f_k, g_k))\Big)^{(2,3)} \\
&= \sum_i ((f_{1i}^{(2)} + f_{2i}^{(2)}, g_i^{(3)}) - (f_{1i}^{(2)}, g_i^{(3)}) - (f_{2i}^{(2)}, g_i^{(3)})) \\
&\quad + \sum_j ((f_j^{(2)}, g_{1j}^{(3)} + g_{2j}^{(3)}) - (f_j^{(2)}, g_{1j}^{(3)}) - (f_j^{(2)}, g_{2j}^{(3)})) + (0,0),
\end{aligned}
$$

implying $[Y_1, Y_2] \otimes [Y_1^2, Y_3] = 0$ in $\mathbb{Z}\{Y\}^2$, easily seen to be false by specializing Y_1 to e_{11}, Y_2 and Y_3 to e_{12}. This contradiction shows $r \neq 0$. Q.E.D.

The same proof shows

$$\mathbb{Z}\{Y\} \underset{C'}{\otimes} \mathbb{Z}(Y) \xrightarrow{\phi \otimes 1} \mathbb{Z}(Y) \underset{C'}{\otimes} \mathbb{Z}(Y)$$

is not injective, so $\mathbb{Z}\{Y\}$ is not C'-flat. Theorem 2 can be used to obtain a negative result for polynomials, not central for $M_n(\Omega')$, whose squares are central for $M_n(\Omega')$.

THEOREM 3. *If 4 divides n and $f(X_1, \cdots, X_n)$ is a polynomial linear in $X_2, \cdots, X_n$ such that f^2 is central for $M_n(\Omega')$, then f is central for $M_n(\Omega')$.*

The proof is long and technical, involving graph theoretic arguments; only the basic idea is given here. Let $Y_1, \cdots, Y_m$, $Y_2', \cdots, Y_m'$ be distinct generic matrices in $\Omega\{Y\}$. Assume f^2 is central for $M_n(\Omega')$. If $f(Y_1, Y_2, \cdots, Y_m)$ and $f(Y_1, Y_2', \cdots, Y_m')$ commute, then (it can be shown) f is central for $M_n(\Omega')$. Hence we may assume $f(Y_1, Y_2, \cdots, Y_m)$ and $f(Y_1, Y_2', \cdots, Y_m')$ do not commute. Let $f_i = f(Y_1, \cdots, Y_i, Y_{i+1}', \cdots, Y_m')$. It follows that f_i and f_{i-1} do not commute for some $i \geqq 2$. But $f_i f_{i-1} - f_{i-1} f_i$ anticommutes with f_i and both these elements have squares in C'; hence they generate a quaternion subalgebra of $\Omega(Y)$. Since $\Omega(Y)$ is a tensor product of any subalgebra and its centralizer, $\Omega(Y)$ has exponent less than n, contradicting Theorem 2. Hence f is indeed central for $M_n(\Omega')$, yielding the theorem.

On the other hand, classical division algebra theory shows that $\Omega(Y)$ has square-central elements if n is of the form $4(2k+1)$, any nonnegative integer k, implying that square-central polynomials do exist for $M_n(\Omega)$, for all such n. Hence linearity in $X_2, \cdots, X_m$ is a crucial condition for Theorem 3 to hold if $8 \nmid n$. An important open question is whether square-central polynomials exist for $M_n(\Omega')$ if $8 \mid n$.

4. Universal PI-algebras over arbitrary Ω

Let Ω be an arbitrary commutative ring, and let $\mathscr{W} = \{T\text{-ideals of } \Omega\{X\}$ containing a power of the standard polynomial$\}$, that is, $W \in \mathscr{W}$ if and only if $\Omega\{X\}/W$ is a universal PI-algebra. Suppose $W \in \mathscr{W}$ and let $\bar{R} = \Omega\{X\}/W$.

PROPOSITION 1. *If $\bar{A} = A/W$ is a T-ideal of $\bar{R}$ then $\bar{R}/\bar{A}$ and $\bar{R}/\text{Ann}\,\bar{A}$ are universal* PI-*algebras (where* Ann $\bar{A}$ *denotes the right annihilator of $\bar{A}$ in $\bar{R}$). In particular, if $\bar{R}$ has nilradical N and Jacobson radical J then $\bar{R}/N$, $\bar{R}/\text{Ann}\,N$, $\bar{R}/J$, and $\bar{R}/\text{Ann}\,J$ are universal* PI-*algebras.*

PROOF. The second assertion follows from the first assertion since the nilradical and Jacobson radical are clearly T-ideals. So assume $\bar{A} = A/W$ is a T-ideal of $\bar{R}$. Clearly A is a T-ideal of $\Omega\{X\}$; moreover if $(S_{2n})^m \in W$ then $(S_{2n})^m \in A$, implying $A \in \mathscr{W}$. Hence $\bar{R}/\bar{A} \approx \Omega\{X\}/A$ is a universal PI-algebra.

Similarly, to prove $\bar{R}/\text{Ann}\,\bar{A}$ is a universal PI-algebra, it suffices to show that $B = \{f \in \Omega\{X\} \mid Af \subseteq W\}$ is a T-ideal of $\Omega\{X\}$; in other words, for any endomorphism ψ of $\Omega\{X\}$ and $f(X_1, \cdots, X_m)$ in B, one must show $\psi(f(X_1, \cdots, X_m)) \in B$. Let $\psi(f(X_1, \cdots, X_m)) = f_1(X_1, \cdots, X_k)$. We may assume $k \geqq m$ (by considering, if necessary, indeterminates occurring trivially in f); we are done if

$$g(X_1, \cdots, X_t) f_1(X_1, \cdots, X_k) \in W$$

for each $g(X_1, \cdots, X_t)$ in A. Well, $g(X_{k+1}, \cdots, X_{k+t}) \in A$ since A is a T-ideal, so $g(X_{k+1}, \cdots, X_{k+t}) f(X_1, \cdots, X_m) \in W$. Define an endomorphism $\psi' : \Omega\{X\} \to \Omega\{X\}$ by $\psi'(X_i) = \psi(X_i)$ for $i \leqq k$, $\psi'(X_i) = X_{i-k}$ for $i > k$. Then $g(X_1, \cdots, X_t)$ $f_1(X_1, \cdots, X_k) = \psi'(g(X_{k+1}, \cdots, X_{k+t}) f(X_1, \cdots, X_m)) \in W$, as desired. Q.E.D.

Let Rad denote the Jacobson radical. Amitsur has proved the next theorem when Ω is an infinite field (refer to [7, Chap. X]).

THEOREM 4. *If U is a universal* PI-*algebra then* Rad U *is nil.*

PROOF. By factoring out the nilradical, it suffices to assume U is semiprime and to prove Rad $U = 0$. Let $J = \text{Rad } U$.

Case I. Every identity of U is the sum of completely homogeneous identities. In this case, it is well known (refer to [10, Prop. 1.3]) that U and $U[\lambda]$ satisfy the same identities, λ a commutative indeterminate. Since $U[\lambda]$ is semiprimitive (a consequence of a theorem of Amitsur in [7, p. 10] since semiprime PI-algebras have no nonzero nil ideals) and since U is universal, we obtain a sequence of surjections $U \to U/J \to U[\lambda]$. But then U/J is a universal PI-algebra satisfying the same identities as U, so $J = 0$, proving the theorem in Case I. (Note this case subsumes Amitsur's result; in fact such a proof has been known by Amitsur.)

Case II. In general, let $\bar{U} = U/\text{Ann}\, J$ and $\bar{J} = (J + \text{Ann}\, J)/\text{Ann}\, J$. $\bar{U}$ is a universal PI-algebra by Theorem 1. Also, $\bar{U}$ is semiprime. (Indeed, suppose there is an ideal A of U with $A^2 \subseteq \text{Ann}\, J$. Then $(JA)^2 \subseteq JA^2 = 0$, implying $JA = 0$; hence $A \subseteq \text{Ann}\, J$, so $\bar{A} = 0$.) Likewise, Ann $\bar{J} = 0$. On the other hand, setting $H = \text{cent } \bar{U}$, we see $\bar{J} \cap H$ is a quasi-regular ideal of H, so that $\text{Ann}(\bar{J} \cap H) = 0$. (Proof: let $\bar{B} = \text{Ann}(\bar{J} \cap H)$. Then $(H \cap \bar{J}\bar{B})^2 \subseteq (H \cap \bar{J})\bar{B} = 0$, so $H \cap \bar{J}\bar{B} = 0$; thus $\bar{J}\bar{B} = 0$ by Theorem A, so $\bar{B} \subseteq \text{Ann } \bar{J} = 0$.) This observation, in conjunction with Case I, reduces the theorem to the following lemma.

LEMMA. *Let R be a semiprime* PI-*algebra with center C, such that* Ann Rad C $=0$. *Then all identities of R are sums of completely homogeneous identities.*

PROOF. Suppose an identity $f(X_1, \cdots, X_m)$ of R is not homogeneous in X_1, and let $f_i(X_1, \cdots, X_m)$ be the sums of those monomials of f with degree i in X_1. Clearly $f(X_1, \cdots, X_m) = \sum_i f_i(X_1, \cdots, X_m)$; we shall prove each f_i is an identity of R, and the lemma will follow by iteration of this procedure on each indeterminate. Choose arbitrarily $r_1, \cdots, r_m$ in R and let $y_i = f_i(r_1, \cdots, r_m)$, $0 \leqq i \leqq d$, where d is the degree of f in the first indeterminate. For any c in Rad C, $0 \leqq j \leqq d$, $\sum_{i=0}^{d} c^{ji} y_i = f(c^j r_1, r_2, \cdots, r_m) = 0$. Using the Vandermonde determinant argument on this system of $d+1$ equations (with y_i as the variables, $0 \leqq i \leqq d$), we obtain $g(c)y_i = 0$ for all i, where $g(c)$ is a product of terms of the form $c^p - c^q$, $p < q$. Let $g(c) = c^t g_1(c)$, g_1 a polynomial in c having constant term 1. Since $c \in \text{Rad } C$, $g_1(c)$ is invertible, so $c^t y_i = 0$ for all i. Thus $(cy_iR)^t = 0$, implying $cy_i = 0$, all i and all c in Rad C. Hence $y_i \in \text{Ann Rad } C = 0$, all i, implying each f_i is an identity of R, as claimed. Q.E.D.

Theorem 4 can be applied to algebras of generic matrices $\Omega\{Y\}$ since these are universal PI-algebras (by Theorem 1).

THEOREM 5. Rad $(\Omega\{Y\})$ *is the set of nilpotent elements of* $\Omega\{Y\}$.

PROOF. In view of Theorem 4, we need only show each nilpotent element of $\Omega\{Y\}$ is in $\mathrm{Rad}(\Omega\{Y\})$. Suppose $f(Y_1,\cdots,Y_m)^t = 0$. Then $f(X_1,\cdots,X_m)^t$ is an identity of $M_n(\Omega')$, hence of $M_n(\Omega'/P)$ for any prime ideal P of Ω'. But Ω'/P is an infinite domain, so $f(X_1,\cdots,X_m)$ is an identity of $M_n(\Omega'/P)$ (by the remarks preceding Theorem 2). If N' is the nilradical of Ω' then $M_n(N')$ is the nilradical of $M_n(\Omega')$ and we conclude $f(Y_1,\cdots,Y_m)\in M_n(N')\cap\Omega\{Y\}\subseteq \mathrm{Rad}(\Omega\{Y\})$. Q.E.D.

COROLLARY. $\Omega\{Y\}$ *has no nonzero nilpotent elements if and only if* Ω *is semiprime.*

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